Pergamon

0042-6989(94)00170-7

Vision Res. Vol. 35, No. 6, pp. 827–835, 1995
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0042-6989/95 $9.50 + 0.00

Proximity Judgments in Color Space: Tests of a Euclidean Color Geometry

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Received 17 November 1993; in revised form 19 May 1994

We describe two tests of the hypothesis that human judgments of the proximity of colors are consistent with a Euclidean geometry on color matching space. The first test uses proximity judgments to measure the angle between any two intersecting lines in color space. Pairwise estimates of the angles between three lines in a plane were made in order to test the additivity of angles. Three different color proximity tasks were considered. Additivity failed for each of the three proximity tasks. Secondly, we tested a prediction concerning the growth of the variability of judgments of similarity with the distance between the test and reference stimuli. The Euclidean hypothesis was also rejected by this test. The results concerning the growth of variability are consistent with the assumption that observers use a city-block metric when judging the proximity of colored lights.

Color geometry   Color similarity   Salience of colors   Euclidean geometry

INTRODUCTION

The symmetric color-matching performance of a trichromatic observer can be summarized by a three-dimensional space ("color space") and a rule that assigns to any light a point in the space. Two lights match precisely when they are assigned the same point. The rule is linear and, as a consequence, any invertible linear transformation of a color space accounts for the observer's performance as well as did the original space. Color space is *affine*: vector addition of vectors in color space mirror the superposition of corresponding lights, but color-matching performance alone does not permit estimation of distances between points in color space or angles between lines in color space. A second color task is needed to psychophysically assess the metric and geometric properties of color space.

It is possible that color space has *no* metric or geometric properties, i.e. that no task allows us to assign a consistent metric or geometry to color space or to speak meaningfully about orthogonal directions in color space. Much research in color vision has sought to establish a metric or coordinate system in color space (or color space transformed in some simple way) that accounts for performance in a psychophysical task. Attempts to develop line-element models to account for color discrimination serve as examples (MacAdam, 1944).

There is a long history of attempts to interpret some psychophysical task as a measure of the proximity of points in color space (Ekman, 1954; von Helmholtz, 1891; Indow, 1980; Krantz, 1967). Multidimensional scaling approaches begin with judgments of the relative "similarity" of pairs of, in general, quite different lights. These data are then scaled to establish the number and nature of the fundamental coordinates of a Euclidean space which best encompass the judgments. It has proven difficult to use multidimensional scaling methods to test whether the Euclidean representation is appropriate or to test whether the Euclidean representation is consistent with the linear structure of color space (Torgerson, 1958; Indow & Kanazawa, 1960; Shepard, 1962; Indow, 1980; Helm, 1964; Carroll & Chang, 1970; Carroll & Wish, 1974).

In addition, such ratings of color similarity are proximity tasks that are potentially dependent on the instructions given to subjects and cultural factors. There are, however, well-defined proximity tasks for which the term "similarity" seems inappropriate and which are possibly immune to cultural factors [e.g. color discrimination (MacAdam, 1944) and reaction time (Mollon & Cavonius, 1986)]. We emphasize that there are potentially many proximity tasks, and it is possible that all, some, or none of them can be accounted for by a single Euclidean metric imposed on color matching space. We will return to this point in the Discussion. We will use the terms "proximity task" and "proximity judgment" to refer to any task that could potentially be accounted for by distances between colors in some Euclidean geometry imposed on color matching space. Throughout the remainder of this paper, "color space" will refer to

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the affine space used to explain color matching in the normal trichromatic observer.

Consider the hypothesis that, for a particular proximity task, there is a Euclidean metric on color space that accounts for judgments of proximity. We refer to this hypothesis as "the Euclidean hypothesis". We introduce a new method which requires only a small number of proximity judgments to measure the angle between any pair of intersecting lines in color space if the Euclidean hypothesis holds. The method can also be used to *test* whether the Euclidean hypothesis holds for a particular proximity task. We use it to test whether any of three different proximity tasks can be accounted for by imposing a Euclidean geometry on color space. None of the three proximity tasks could be accounted for by a Euclidean geometry imposed on color space.

In a second test of the Euclidean hypothesis, we tested a prediction concerning the variability of judgments of similarity with the distance between the test and reference stimuli. The variability of the judgments failed to grow with distance as might plausibly be expected from Euclidean geometry.

Geometric idea

In this section and the next we describe a method that uses proximity judgments to test whether an affine space admits a Euclidean geometry. If the affine space does have a Euclidean geometry that is consistent with the proximity judgments, the method permits us to estimate angles between pairs of lines in a color matching space. The method only assumes that the observer can judge the relative proximity of pairs of stimuli and tests whether the proximity judgments are consistent with a Euclidean metric.

We begin by describing how to estimate the angle between any two intersecting lines. Consider two lines Λ_1 and Λ_2 in a plane in a Euclidean space [Fig. 1(b)]. Figure 1(b) is a hypothetical internal color space consistent with a Euclidean geometry. Suppose stimulus A_1 is fixed on line Λ_1. The observer then considers the stimuli corresponding to the line Λ_2 and judges the stimulus on Λ_2 that is most proximate to A_1. This most proximate stimulus is denoted by B_2. In the Euclidean space, there must be a unique nearest point on the line, and the line joining the stimuli A_1 and B_2 must be orthogonal to the line Λ_2 in Fig. 1(b). We term this procedure an *orthogonality judgment*.

Figure 1(a) represents the experimenter's space (e.g. the MacLeod–Boynton space) which is some unknown linear transformation of the internal color space (the linear function Ψ maps the experimenter's space into color space). The experimenter can choose the light a_1 which is mapped to A_1. He also knows the light b_2 corresponding to B_2, the light on Λ_2 most proximate to A_1. The lines λ_1 and λ_2 are mapped to the lines Λ_1 and Λ_2, respectively. Note that, in the experimenter's space, the line $a_1 b_2$ will typically not be perpendicular to λ_2. The angle Θ is the angle between Λ_1 and Λ_2 in color space. There is no corresponding marked angle in the experimenter's space to emphasize that the experimenter's space is affine.

A single orthogonality judgment cannot determine the angle Θ between the lines Λ_1 and Λ_2. Not only is the angle unknown, but also the relative scaling along the two lines induced by the unknown linear transformation Ψ that carries Fig. 1(a) into Fig. 1(b). With two such judgments, properly chosen, it is possible to determine the angle itself.

Corresponding points and lines in the two figures are shown in lower case [Fig. 1(a)] and upper case [Fig. 1(b)]. Figure 1(a) illustrates (from the experimenter's point of view), two point-to-line proximity judgments of the sort just described: b_2 is the point on λ_2 most proximate to a_1 fixed, on λ_1. Similarly, b_1 is the point on λ_1 most proximate to a_2 fixed, on λ_2.

Note that the distances OA_1, OA_2, OB_1, and OB_2, are all unknown to the experimenter.

$$\cos(\Theta) = \frac{OB_2}{OA_1} = \frac{OB_1}{OA_2}. \tag{1}$$

The quantities on the right-hand side are unknown. The distances oa_1, oa_2, ob_1, and ob_2, are all known to the experimenter. As a consequence of the linearity of the transformation from Fig. 1(a) to Fig. 1(b),

$$\frac{ob_1}{oa_1} = \frac{OB_1}{OA_1} \tag{2}$$

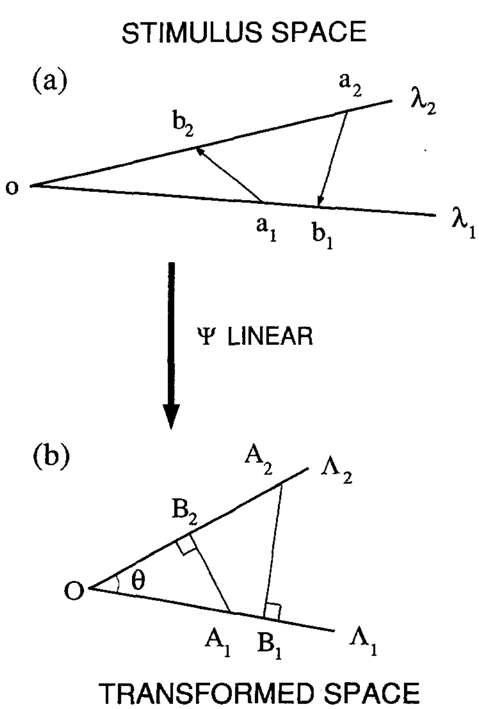

FIGURE 1. Stimulus space (e.g. MacLeod–Boynton space) and transformed space (internal representation). (a) The stimuli available to the experimenter. Each point in (a) corresponds to a light used in a color proximity task. The lines λ_i in the experimenter's space are formed by mixing lights. (b) The observer's internal color space. The mapping Ψ between the two spaces is assumed to be linear. Each lower-case point or line in (a) is mapped to the corresponding upper case point or line. See text for a description of the marked points and lines.

and

$$\frac{ob_2}{oa_2} = \frac{OB_2}{OA_2}. \quad (3)$$

(The linear transformation Ψ preserves the ratio of the lengths of collinear line segments.) From equation (1),

$$\cos^2(\Theta) = \frac{OB_2}{OA_1}\frac{OB_1}{OA_2} = \frac{OB_2}{OA_2}\frac{OB_1}{OA_1} = \frac{ob_2}{oa_2}\frac{ob_1}{oa_1} \quad (4)$$

and we can now compute the cosine squared of the unknown angle in terms of quantities known to the experimenter.

It is also possible to determine the cosine and, therefore, the (unsigned) angle Θ between the two lines as follows. The points o, a_1 and b_1 are collinear and, if they are distinct, one is between the other two (on the line λ_1). The point o is between a_1 and b_1 precisely when the point O is between A_1 and B_1. The same is true for the points on the lines λ_2 and Λ_2. If Θ is >90 deg, then the point o will lie between a_1 and b_1 on λ_1 and also between a_2 and b_2 on λ_2. If Θ is <90 deg, then the point o will not lie between a_1 and b_1 on λ_1 and will not lie between a_2 and b_2 on λ_2. (It is not possible for the point o to lie between the two points on one line but not the other.) We can therefore determine the magnitude of Θ between 0 and 180 deg by examining the betweenness relations of the points o, a_i, b_i, $i = 1, 2$. The sign of Θ tells us whether the angle runs clockwise or counterclockwise and cannot be determined from these data.

Two orthogonality judgments based on proximity judgments determine the angle Θ. It should be noted that the derivation of the angle does not depend on prior knowledge of the transformation Ψ. We would get the same estimate Θ for the angles in Fig. 1(b) even if we transformed Fig. 1(a) linearly before doing the computation above. It is only assumed that Ψ is linear. In a plane, the transformation Ψ can be determined up to a rotation/reflection and a common scale factor from two orthogonality judgments. For three dimensions, five orthogonality judgments between all pairs of three lines suffice to determine the transformation Ψ.

The method is based on the following two observations, (1) that judgments of orthogonality of lines in a normed linear space determine the space up to a similarity transformation (rotation/reflection, and overall scaling) (see Suppes, Krantz, Luce & Tversky, 1989, p. 31ff); (2) that proximity judgments in a Euclidean space suffice to determine when intersecting lines are orthogonal (Young, 1988, Section 3.2).

Test of consistency

If proximity measurements are controlled by an underlying metric in a Euclidean space, we can determine the angles between lines and the metric of the space up to an unknown scale factor. Suppose that, for any three *co-planar* lines in the space λ_1, λ_2, and λ_3, we measure the angle Θ_{12} between λ_1 and λ_2, the angle Θ_{23} between λ_2 and λ_3, and the angle Θ_{13} between λ_1 and λ_3. If proximity measurements are controlled by an underlying Euclidean space that is a linear transformation of the stimulus space ("The Euclidean Hypothesis"), then $\Theta_{13} = \Theta_{12} + \Theta_{23}$ necessarily. We can therefore test the Euclidean hypothesis by testing additivity of the angles between all pairs of three co-planar lines by consideration of only proximity judgments. We next test the method on three different kinds of color proximity judgment.

METHODS

The stimuli were displayed on a Barco Calibrator television monitor operating at 120 frames/sec, interlaced. The mean luminance of the display was 35 cd m^{-2}. The mean chromaticity was that of an equal-energy white, i.e. the chromaticity was $x = 0.33$, $y = 0.33$ in CIE coordinates. The resolution was 1 min arc per pixel at the viewing distance of 1.75 m. The experiments were run under computer control using the Postq experimental operating system.

The chromatic output of the monitor was frequently calibrated and found to be consistent within about 1%. The stimuli were initially specified using the axes proposed by MacLeod and Boynton (1979) and shown to have singular properties by Krauskopf, Williams and Heeley (1982). The axes are labeled $L - 2M$ and $S - (L + M)$ to specify the weights assigned to the different cone classes by the mechanisms which respond uniquely to stimuli that vary along these axes. The third, luminance, axis, along which stimuli did not vary in these experiments, would be labeled $L + M$ to be consistent. We could have labeled the axes in terms of the variation in cone inputs of stimuli varying along these axes. In this case the axes we call $L - 2M$, $S - (L + M)$ and $L + M$ would be labeled $L - M$, S and $2L + M + S$, respectively.

Common procedures

In all the experiments the stimuli were presented as Gaussian pulses in time with a sigma of 160 msec and a total duration of 1000 msec. They appeared on a square background 512 min arc on a side. The different configurations used are illustrated in Fig. 2. In each case the observer was asked to choose which of two stimuli b and c appeared more similar to a. In each session the stimulus a on line λ_1 was kept constant. Three different standards (b) on line λ_2 were used. Each standard was paired with nine different equally-spaced (in cone space) comparison stimuli (c). Each of these 27 pairs was presented 10 times in a randomized order. The standard and comparison lights were constrained to lie on a line which did not include a.

Spatial configurations and specific tasks

In task A two 2-deg disks b and c were superimposed on a 6-deg disk [Fig. 2(a)] The observers were instructed to judge which of the two small disks is "least salient" when presented on the big disk. In task B the three colors are arranged in a circle [Fig. 2(b)]. In this case the observers were instructed to judge which of the two small wedges "completes the disk more readily". In task C the

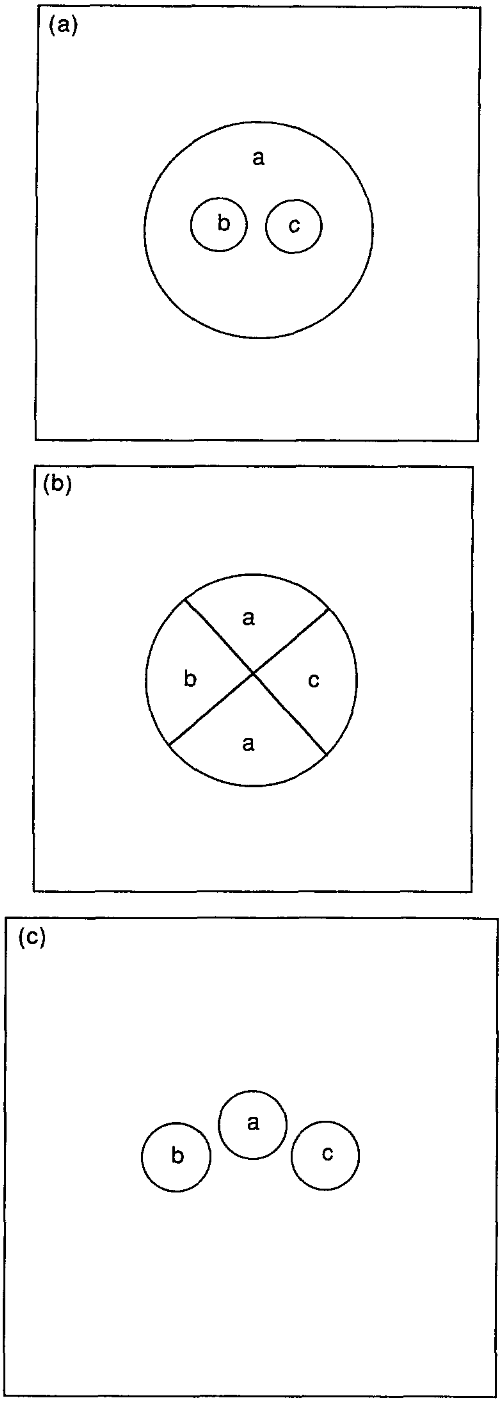

FIGURE 2. Stimulus configurations. Test *a*, comparisons *b* and *c*. (a) *b* and *c*, 2 deg in diameter, *a*, 6 deg in diameter. (b) Circle 6 deg in diameter. (c) Stimuli 2 deg in diameter.

three colored lights are not adjacent [Fig. 2(c)]. In this case the observer was instructed to judge which of the disks *b* or *c* were "more similar in color" to the disk *a*.

Estimation procedures

A psychometric function was plotted for each standard *b* depicting the frequency with which *c* was judged more proximate than *b* to *a*, as a function of the distance of *c* from *b* (Fig. 3). If the standard *b* coincides with the most proximate point then we expect to find a maximum relative frequency of 0.5 at $c = b$ [Fig. 3(a)]. If the most proximate point is different from the standard *b* then we expect to find a maximum relative frequency >0.5 at $c \neq b$ [Fig. 3(b)]. In either case, the location of the maximum relative frequency denotes the most proximate point which was estimated from the mean of a Gaussian curve fitted to the results. In each session three estimates of the most proximate point are obtained (one estimate per standard). Each condition is run three times, hence resulting in nine estimates of the most proximate point for each condition.

Stimuli for additivity test

The MacLeod–Boynton color space representation of the stimuli used in this experiment is given in Fig. 4. For each of the test lights a_1, a_2, and a_3 the most proximate point on the two respective other lines was determined. That is, a_1 on λ_1 was fixed and the most proximate point on the lines λ_2 and λ_3 was determined experimentally. The most proximate points for a_2 and a_3 were found in a similar fashion. All lights were of equal luminance. The test light a_3 along the $L - 2M$ line (λ_3) was chosen at one-tenth of the gamut available; the test light along the $S - (L + M)$ line was at one-fifth of the available gamut. The incremental cone coordinates (ΔL, ΔM, ΔS) of the three test stimuli with respect to the white point ($L = 0.66$, $M = 0.34$, $S = 0.017$) were as follows: for a_1, $\Delta L = 0$, $\Delta M = 0$, $\Delta S = -0.0029$; for a_2, $\Delta L = 0.0028$, $\Delta M = -0.0028$, $\Delta S = -0.0014$; for a_3, $\Delta L = 0.0056$, $\Delta M = -0.0056$, $\Delta S = 0$.

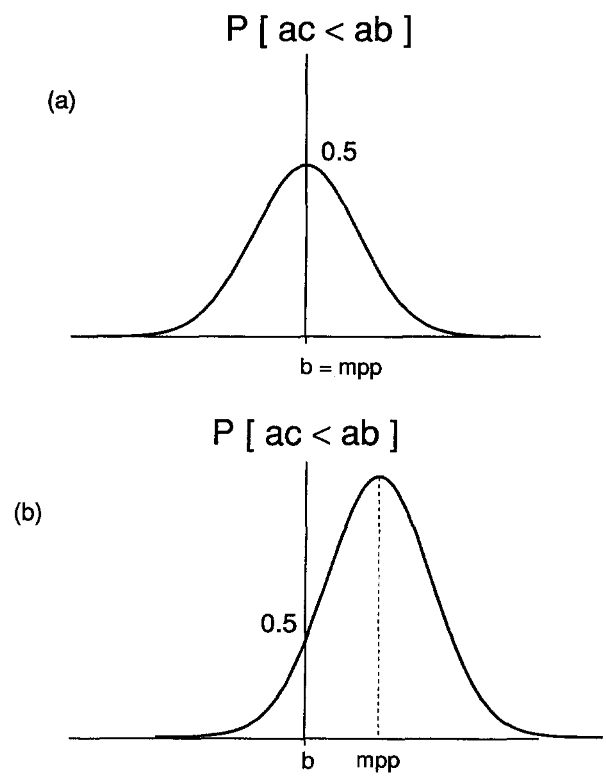

FIGURE 3. Psychometric function obtained: $P[ac < ab]$ is the frequency with which the light *c* is judged as more proximate to *a* than light *b* is. (a) When standard *b* coincides with the most proximate point. (b) When the most proximate point differs from the standard.

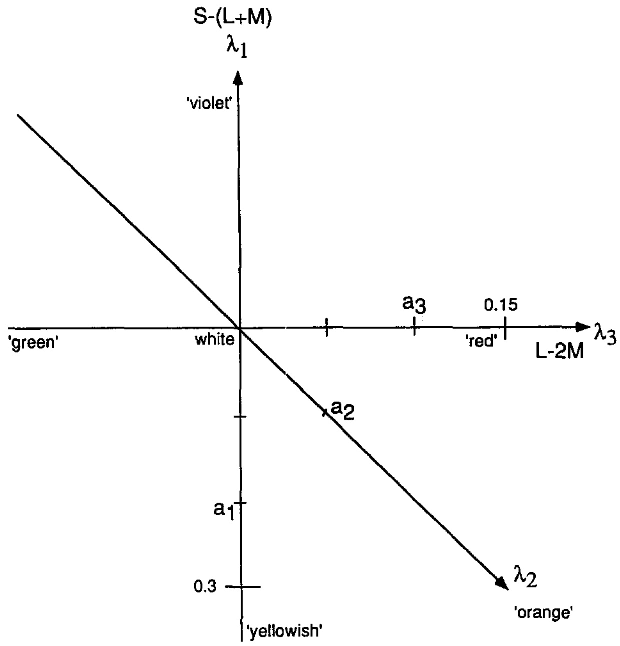

FIGURE 4. Test of additivity of angles. MacLeod–Boynton color space. Test lights are denoted by a_1, a_2, and a_3. For each of these test lights the most proximate point on the other two lines was found experimentally. From these most proximate points the transformations of Θ_{12}, Θ_{23} and Θ_{13} were derived.

Stimuli for variability test

The stimuli were arrayed along two pairs of lines in an equiluminant plane in color space. One pair of lines was identical to that used in the previous experiment (Fig. 4), the $L - 2M$ line (λ_3) and the $S - (L + M)$ line (λ_1). The other pair of lines were lines intermediate between these two lines, that is a line ranging from the background white to orange (λ_4), and a line from the background white to purple (λ_5).

The most intense test stimulus on the $L - 2M$ line produced a cone contrast of 0.033 in the L cones, 0.066 in the M cones, and 0 in the S cones. Along the $S - (L + M)$ line the maximum test light yielded an S-cone contrast of 0.68 and a cone contrast of 0 in the L and M cones. The most intense stimuli along the orange and the purple line (λ_4 and λ_5) resulted in 0.02 L-cone contrast, 0.04 M-cone contrast, and 0.68 S-cone contrast.

Subjects

Four subjects participated in the experiment. Subject RP was naive as to the purpose of the experiment, subject JM was an experienced observer and aware of the purpose of the experiment. Subjects LM and SW are authors of this paper.

RESULTS

Additivity of angles

The first issue addressed was the additivity of angles employing the stimuli shown in Fig. 4. We test whether the angle Θ_{13} (angle between Λ_1 and Λ_3) is equal to the sum of Θ_{12} and Θ_{23}. These angles are defined in the transformed space [see also Fig. 1(b)]. The values of these angles, Θ_{12}, Θ_{23}, Θ_{13}, were measured using the methods detailed above. Table 1 shows for all four subjects and for all three tasks the mean and the standard error (in parentheses) of the derived angles (Θ_{12}, Θ_{23}, Θ_{13}). In all cases, the angle Θ_{13} (angle between Λ_1 and Λ_3) is smaller than the sum of the angles Θ_{12} (angle between Λ_1 and Λ_2) and Θ_{23} (angle between Λ_2 and Λ_3). The sum $\Theta_{12} + \Theta_{23}$ exceeds Θ_{13} by approx. 15%. The null hypothesis that $\Theta_{12} + \Theta_{23} = \Theta_{13}$ may be rejected for all four subjects at a level of significance of $\alpha < 0.005$ on the basis of a contrast test proposed by Scheffe (1959, p. 66).

For each task the subject made at least nine separate estimates of each of the three angles, Θ_{12}, Θ_{23}, Θ_{13}; each estimate was based on 10 repetitions. For all settings and for each subject, $\Theta_{12} + \Theta_{23} > \Theta_{13}$. The means and standard deviations in Table 1 illustrate the agreement between and within subjects in the angles estimated.

For both task A and task B, the average angle derived from the projections of a_1 on Λ_3 and a_3 on Λ_1 is approx. 90 deg (Table 1). A different result is found for task C: the angle between Λ_1 and Λ_3 is approx. 140 deg for observer SW, that is, the yellowish-greenish light on the Λ_1 line is perceived as much closer to the greenish light on Λ_3 than to a gray light on the same line. Observers clearly judge different aspects of the colored lights given the different instructions and spatial configurations.

Effect of distance on variability

There are several reasons to expect that the variability of the judgments made in these experiments would increase with increase in the distance between the standard stimulus (a) and the comparison stimuli (b and c). Among them is the specific hypothesis that the observers base their judgments on the Euclidean distances between internal representations of the stimuli.

The experimental situation is illustrated in Fig. 5, in which it is assumed that Λ_1 is orthogonal to Λ_2 in the transformed space and that the lines intersect at B. The assumption is made that the probability of judging C to be more proximate to A than B is a function of the excess distance (D) of C from A compared to that between A and B so that, for example, σ (see Fig. 3) will vary in proportion to the distance between B and C. Since B is always the zero point and A and C are points with one

TABLE 1

Subject	Angle Θ_{12}	Angle Θ_{23}	Angle Θ_{13}	$\Theta_{12} + \Theta_{23} - \Theta_{13}$
Additivity of angles: task A				
Subject JM	50.4 (4.7)	61.0 (1.7)	87.6 (2.9)	23.8
Subject LM	52.0 (3.1)	55.9 (2.9)	85.4 (4.3)	22.5
Subject RP	39.4 (5.1)	64.3 (4.2)	89.9 (2.9)	13.8
Subject SW	47.9 (3.9)	56.6 (5.3)	88.1 (4.5)	16.4
Additivity of angles: task B				
Subject RP	37.8 (9.2)	67.4 (4.2)	94.4 (5.7)	
Additivity of angles: task C				
Subject SW	78.0 (9.8)	82.3 (9.4)	137.0 (10.5)	

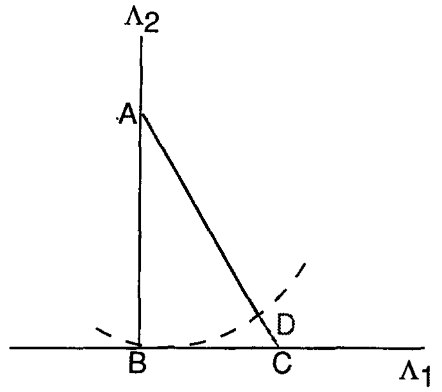

FIGURE 5. Typical set of stimuli used in experiments on the effect on variability of distance between test and comparison stimuli. Under a Euclidean model the observer computes the excess distance D to judge whether A is closer to B or closer to C.

coordinate being zero, we can use the same symbol to denote a point and its length without introducing any essential ambiguity. From Fig. 5 it can be deduced that:

$$A^2 + C^2 = (A + D)^2 \tag{5}$$

and therefore:

$$C = (2AD + D^2)^{1/2}. \tag{6}$$

Since $C = D$ when $A = 0$ we can determine the values of D from a threshold detection experiment. To normalize the results we assign a value of 1 to D and then equation (6) reduces to:

$$C = (2A + 1)^{1/2}. \tag{7}$$

We used task A to test the variability prediction. In preliminary experiments, the lines λ_4 and λ_5 were chosen to be approximately orthogonal in the transformed space. This is important because the prediction of the variation of the variability is easier to derive for orthogonal stimuli. That the stimuli used met this criterion well is supported by the plot (Fig. 6) of the mean position judged to be most proximate to the test stimulus as a function of the distance between the test stimulus and the comparison stimuli. The mean is close to zero and almost never departs by 2 SE from zero. Error bars denote 2 SDs of the mean.

The variation of σ as a function of distance between the test and comparison stimuli is presented in Fig. 7 along with the prediction just derived. The results are presented in normalized units where the normalizing value was determined in a detection experiment. The different symbols refer to the four different lines used in this experiment. For instance, if the test stimuli were on the $L - 2M$ line (λ_3), then the variability was measured along the $S - (L + M)$ line (λ_1) (see Fig. 4); if the test stimuli were on the $S - (L + M)$ line, then the variability was measured along the $L - 2M$ line. These data are denoted by circles. Squares refer to the case, where the test lights are either on the orange line or on the purple line.

The variability is nearly constant and does not differ reliably from 1 for all distances between the test and comparison stimuli. Error bars denote 2 SEs. It is clear that the prediction based on equation (7) must be rejected.

DISCUSSION

Two fundamental tests were applied to the Euclidean hypothesis and it failed both. The additivity of angles prediction failed for all the tests we made. As mentioned above, there are several reasons why an increase of variability might be expected as a function of the distance between the test and comparison stimuli. Equation (7) is derived on the assumption that the probability of correct judgments of relative distance is a function only of the excess distance between the test and each of the comparison stimuli. On the assumption that

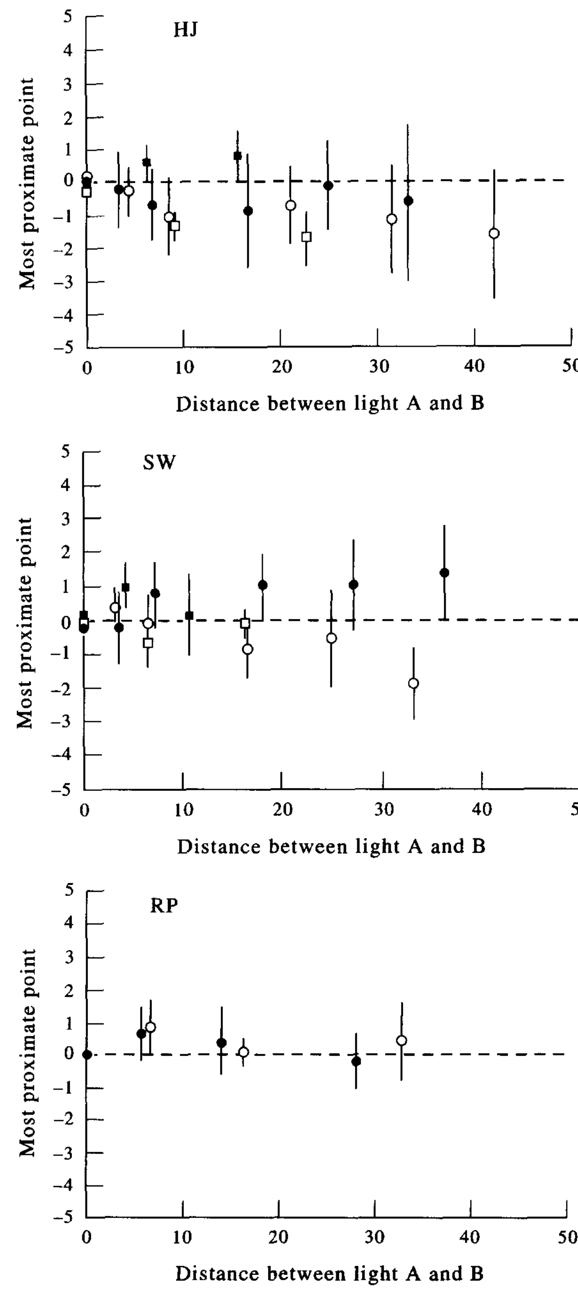

FIGURE 6. Test of orthogonality. Variation of location of most proximate stimulus with distance between test and comparison stimuli.

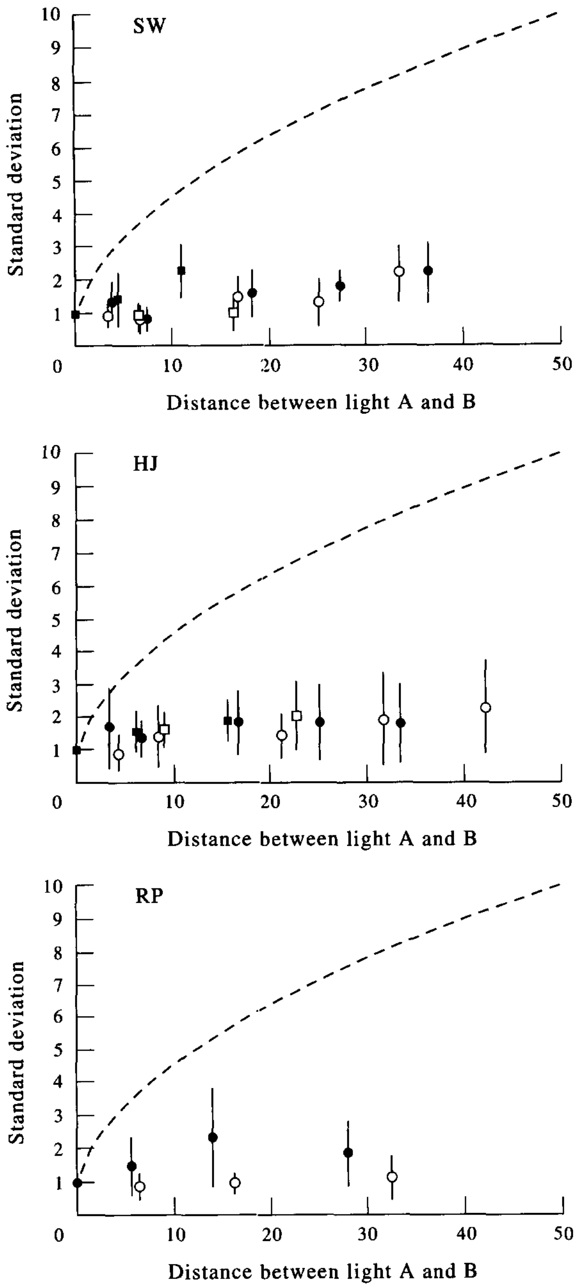

FIGURE 7. Variability test. Variation of σ of psychometric function with distance between test and comparison stimuli.

the error in estimating the magnitude of each of these distances grows with distance (Weber-law behavior) we would expect an even larger variance with increased distance. We can think of no countervailing mechanism for reducing error with increase in distance.

Both experiments reported here produced results that are inconsistent with the notion that people make proximity judgments on the basis of the Euclidean distance between internal representations of the stimuli. Four possible reasons for this failure are (1) that we used unsuitable stimulus conditions and/or an unsuitable proximity task; (2) that people simply do not use a Euclidean representation of chromatic stimuli when making proximity judgments but do use some metric consistent with the linear structure of color space; (3) that people use a Euclidean representation, but that the representation is not consistent with the linear structure of color space (the lines and planes in color matching space are nonlinear curves in the Euclidean color space that controls proximity judgments in a particular task); or (4) no proximity task is consistent with any Euclidean geometry consistent or inconsistent with color space. We discuss the first three reasons next.

(1) There is merit in the first alternative. We cannot rule out the possibility that there are stimulus conditions and a proximity judgment which would pass the tests above. The third arrangement [Fig. 2(c)] is the most similar to traditional methods of stimulus presentation in similarity experiments. It has, however, a feature that may often be overlooked, namely that the relation between the test stimuli and the background may influence the judgment.

(2) The failure to obtain an increase in σ with increased separation between the test and comparison stimuli could be explained if we abandoned the Euclidean metric (a Minkowski ρ-metric of power 2) in favor of a city-block metric. The Euclidean and city-block metrics are both Minkowski ρ-metrics:

$$d(x, y) = \left(\sum_{i=1}^{3} (x_i - y_i)^{\rho} \right)^{\frac{1}{\rho}}.$$

The Euclidean metric obtains when $\rho = 2$, the city-block metric when $\rho = 1$. The Minkowski ρ-metric is a metric for any choice of $\rho \geqslant 1$.

The city-block distance between two points is the sum of their distances along each coordinate axis. The orthogonality tasks of Expt 1 are still well-defined with Minkowski ρ-metric with $\rho \neq 2$. For $\rho = 1$, there are choices of point and line such that the nearest point on the line is not unique. For all other values of ρ, there is always a unique nearest point on a line to a given point not on the line. When ρ is not 2 however, the orthogonality judgments do not permit measurement of the angles between pairs of lines. Indeed, using the orthogonality judgment, it is possible to conclude that a line μ is orthogonal to a line v, but, reversing the roles of the two lines, that v is not orthogonal to μ (Suppes *et al.*, 1989, p. 45).

The data of Expt 1 are consistent with assuming that the observer employs a city-block metric to judge the proximity of colored lights. In other words, the data are consistent with the hypothesis that the observer analyses the colored stimuli in terms of their coordinates along the $L - 2M$ and the $S - (L + M)$ axis, and the overall distance between two lights is simply the sum of the absolute changes along these two axes. This combination rule predicts that the variability is independent of the location of A. However, from our data we cannot conclude that the stimuli are *necessarily* analyzed with respect to these color directions. In fact, it is the case, that a constant variability is predicted for *any basis* [not only $L - 2M$ and $S - (L + M)$] as long as the light A is sufficiently further apart from light B than light C is. Only in cases where A is closer to B than C is, a dependence of the variability on the location of A is predicted by a city-block metric. Since A was chosen to be much more intense than C, our experiment does not allow us to decide whether the color directions we happened to choose are indeed the underlying mechanisms. To decide whether the $L - 2M$ and $S - (L + M)$ lines constitute a distinguished basis for this task (i.e. constitute "mechanisms") more data need to be collected for lights A close to B. Krauskopf and Gegenfurtner (1992) found evidence that these directions are indeed mechanisms for color discrimination.

(3) Proximity judgments can be scaled by an MDS procedure to determine the Euclidean space that best accounts for the judgments. The transformation from color space to the resulting Euclidean space is, in general, nonlinear and consequently lines and planes in color space are curved in the resulting Euclidean space. It would be desirable, for example, to derive a Euclidean space, in which distance corresponded to color similarity, and hue, saturation and brightness codes were linear. As sets of lights adjusted to be equibright are typically not coplanar in color space (Wyszecki & Stiles, 1982, pp. 413–420), this space could not be a linear transformation of color space.

The results of Izmailov and Sokolov (1991) suggest that there is no such space. Consider the "brightness code" in one of Izmailov and Sokolov's experiments, subjects adjusted spectrally-narrowband lights to be equally bright. The equibright, spectrally-narrowband stimuli were then rated for similarity and the numerical ratings interpreted as proximity data. A multidimensional scaling procedure was used to derive two- and three-dimensional solutions for the proximity judgments. Izmailov and Sokolov report that "the minimal dimensionality of the Euclidean space for equibright color discriminations is three". Equally-bright lights are not coplanar in a Euclidean space constructed to be consistent with judgments of color similarity. That is, the brightness code is not linear in such a space.

Since the brightness code is not linear with respect to a similarity-derived Euclidean space, the nonlinearity of the brightness code (or by analogy, hue and saturation codes) with respect to color space does not exclude the Euclidean hypothesis for color space for color similarity judgments or for any other proximity task. Our subjects were never instructed to judge hue or saturation. It is possible that these nonlinear color codes like hue, saturation, and brightness (Burns, Elsner, Pokorny & Smith, 1984) are not the codes used when the observer is asked to judge proximity as defined by task A and B. However, in task C the observer clearly judged hue.

We note that the results of Izmailov and Sokolov are consistent with ours. We chose stimuli that *were* coplanar in color space, not equally-bright. The Izmailov and Sokolov results illustrate that multidimensional scaling is a powerful and under-utilized technique for estimating a spatial representation of stimuli given a measure of proximity of the stimuli. It does not, however, lend itself to tests of the Euclidean assumptions underlying it.

We directly tested linearity for the proximity judgment defined by task A for two lines intermediate between λ_1 and λ_3. That is, for several fixed lights on one line, we found the most proximate point on the other line. The range of the fixed lights was comparable to that used in the main experiment (see Fig. 4). Linearity implies that all the projections from one line onto the other line are parallel. That is, if the fixed light (a) on one line is scaled by a certain amount then the corresponding most proximate point (b) on the other line must be scaled by the same factor. Hence, if the transformation is linear, then the most proximate points (b) are predicted to lie on a straight line when plotted as a function of the fixed light (a). If the transformation from the cone space to the space defined by the proximity judgment is indeed nonlinear then we would expect to observe nonparallel projections, i.e. the most proximate points (b) do not lie on a straight line when plotted vs the fixed lights (a). The data for two subjects (SW and JM) are shown in Fig. 8. The most proximate points (b) are plotted as a function of the fixed lights (a) for two lines intermediate between λ_1 and λ_3. Open and solid circles depict the projections of the two lines onto each other, respectively. The most proximate points fall on approximate straight lines. Hence we conclude that the failure of a Euclidean model is not due to a nonlinear relationship between the cone space and the space defined by the proximity judgments for task A. Rather, we must conclude that observers do

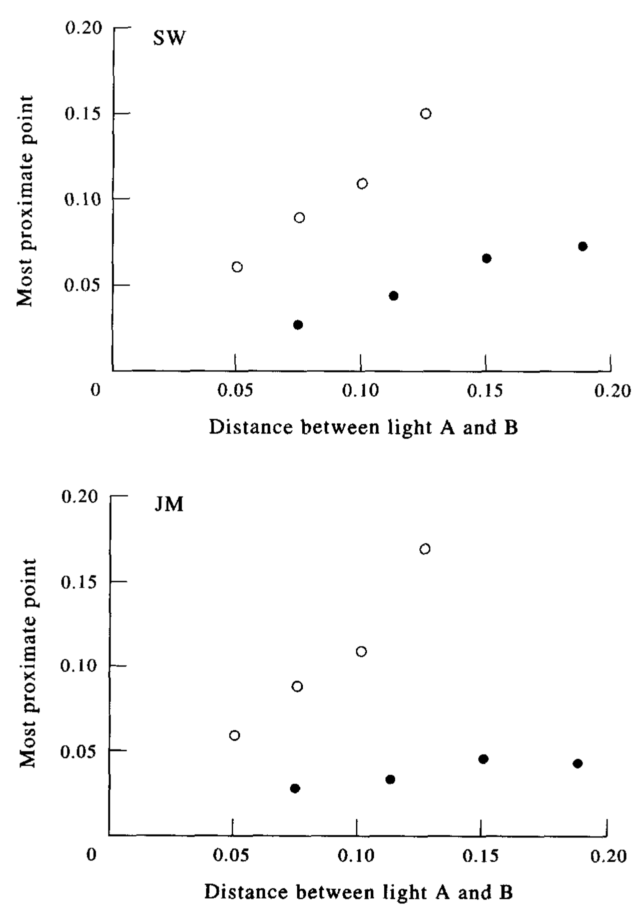

FIGURE 8. Test of linearity. A test of the linearity of the transformation Ψ between the experimenter's space and color space. Linearity implies that the solid and open circles lie on straight lines, respectively. See text for details.

not employ a Euclidean distance measure when judging the proximity of colored lights.

The approach taken here has the merit of greatly simplifying the experimental task of deciding whether any candidate proximity task is consistent with Euclidean geometry on a linear space such as color space. The observer is only required to judge proximity: it is only necessary to decide which of two stimuli are more like a third. Only a small number of judgments are needed to estimate angles. The method is applicable to spaces that are nonlinear transformations of color space. It only requires that the experimenter be able to generate the stimuli that lie on a line in such a space. The test of angle additivity and the test of increasing variability test the Euclidean hypothesis in addition to estimating the geometry and metric in the space.

REFERENCES

Burns, S. A., Elsner, A. E., Pokorny, J. & Smith, V. C. (1984). The Abney Effect: Chromaticity coordinates of unique and other constant hues. *Vision Research*, *24*, 479–489.

Carroll, J. D. & Chang, J. J. (1970). Reanalysis of some color data of Helm's by INDSCAL procedure for individual differences in multidimensional scaling. *Proceedings of the 78th Annual Convention, American Psychological Association* (pp. 137–140).

Carroll, J. D. & Wish, M. (1974). Models and methods for three-way multidimensional scaling. In Krantz, D. H. *et al.* (Eds), *Contemporary development in mathematical psychology, Vol. II: Measurement, psychophysics, and neural information processing* (pp. 57–105).

Ekman, G. (1954). Dimensions of color vision. *Journal of Psychology*, *38*, 467–474.

Helm, C. E. (1964). Multidimensional ratio scaling analysis of perceived color relations. *Journal of the Optical Society*, *54*, 256–262.

von Helmholtz, H. (1891). Kürzeste Linien im Farbensystem. *Sitzungsbericht der Akademie zu Berlin*, 109–122.

Indow, T. (1980). Global color metric and color-appearance systems. *COLOR Research and Application*, *5*, 5–12.

Indow, T. & Kanazawa (1960). Multidimensional mapping of Munsell colors varying in hue, chroma, and value. *Journal of Experimental Psychology*, *59*, 330–336.

Izmailov, Ch. A. & Skolov, E. N. (1991). Spherical model of color and brightness discrimination. *Psychological Science*, *2*, 249–259.

Krantz, D. (1967). Small-step and large-step color differences for monochromatic stimuli of constant brightness. *Journal of the Optical Society*, *57*, 1304–1316.

Krauskopf, J. & Gegenfurtner, K. R. (1992). Chromatic discrimination and adaptation. *Vision Research*, *32*, 2165–2175.

Krauskopf, J., Williams, D. R. & Heeley, D. M. (1982). Cardinal directions of color space. *Vision Research*, *22*, 1123–1131.

MacAdam, D. L. (1944). On the geometry of color space. *Journal of the Franklin Institute*, *238*, 195.

MacLeod, D. I. A. & Boynton, R. M. (1979). Chromaticity diagram showing cone excitation by stimuli of equal luminance. *Journal of the Optical Society of America A*, *69*, 1183–1186.

Mollon, J. D. & Cavonius, C. R. (1986). The discriminability of colours on c.r.t displays. *Journal of the Institution of Electronic and Radio Engineers*, *56*, 107–110.

Scheffe, H. (1959). *The analysis of variance*. New York: Wiley.

Shepard, R. N. (1962). The Analysis of proximities: Multidimensional scaling with an unknown distance function. *Psychometrika*, *27*, 125–140 and 219–246.

Suppes, P., Krantz, D. H., Luce, R. D. & Tversky, A. (1989). *Foundations of measurement. Vol. II: Geometrical, threshold, and probabilistic representations*. New York: Academic Press.

Torgerson, W. S. (1958). *Theory and method of scaling*. New York: Wiley.

Young, N. (1988). *An introduction to Hilbert space*. Cambridge: Cambridge University Press.

Wyszecki, G. & Stiles, W. S. (1982). *Color science: Concepts and methods, quantitative data and formulae* (2nd edn). New York: Wiley.

Acknowledgements—The experiments were performed using the Postq experiment control system and a modified version of the Vistaboard display device designed by Walter Kropfl. We are grateful to Michael Landy, Kenneth Knoblauch, and Professor Drösler for their helpful comments. The research described in this paper was supported in part by DFG grant Wu 204/1-1 from the German Research Foundation to Sophie Wuerger, AFOSR grant F49620-92-J-0187 to Laurence T. Maloney, and National Eye Institute grant EY06638 to John Krauskopf. Some of this work was presented at the annual meeting of the Association for Research in Vision and Ophthalmology, Sarasota, Fla, May 1993 and at the ECVP at Edinburgh, August 1993.